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Commutators for Approximation Spaces and Marcinkiewicz-Type Multipliers*

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It is proved that, under some conditions, weaker than those of the Marcinkiewicz multiplier theorem, the multiplier operator $T_\mu(\sum_k c_k e^{ikt}) = \sum_k \mu_k c_k e^{ikt}$ satisfies on the Besov space $B_p^{\sigma, q}$ the commutator theorem

$$\|[T, T_\mu]\|_{B_p^{\sigma, q}, B_p^{\sigma, q}} \leqslant c\, \|T\|,$$

where $\|T\| = \max(\|T\|_{B_p^{\sigma_0, q_0}, B_p^{\sigma_0, q_0}}, \|T\|_{B_p^{\sigma_1, q_1}, B_p^{\sigma_1, q_1}})$ and $\sigma_0 > \sigma > \sigma_1 > 0$. © 1999 Academic Press

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1. INTRODUCTION

The main idea underlying this paper is that the description of approximation spaces and the calculation of almost optimal approximation elements, in combination with real interpolation, are very useful in the so-called commutator theorems.

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Let $\bar{X} = (X_0, X_1)$ be a Banach couple and, for any $x \in \Sigma(\bar{X}) = X_0 + X_1$ and $t > 0$, let us denote

$$K(t, x) = K(t, x; \bar{X}) = \inf_{x = x_0 + x_1} \{ \|x_0\|_{X_0} + t \, \|x_1\|_{X_1} \},$$

the Peetre's K–functional. If $0 < \theta < 1$ and $1 \leqslant q \leqslant \infty$, we denote $(\bar{X})_{\theta, q}$ the corresponding interpolation space defined by the real K-method, endowed with the norm

$$\|x\|_{\theta, q} = \|t^{-\theta} K(t, x)\|_{L_q(dt/t)},$$

and $\mathscr{L}(\bar{X}; \bar{Y})$ the vector space of all linear operators $T : \Sigma(\bar{X}) \to \Sigma(\bar{Y})$ such that $T(X_j) \subset Y_j (j = 0, 1)$ and $\|T\| = \max(\|T\|_{X_0, Y_0}, \|T\|_{X_1, Y_1}) < \infty$.

If

$$Sf(t) = \int_0^t f(s) \frac{ds}{s} + t \int_t^\infty f(s) \frac{ds}{s^2} = \int_0^\infty f(s) \min(1, t/s) \frac{ds}{s}$$

is the Calderón operator, we set

$$\sigma(\bar{X}) = \{ x \in \Sigma(\bar{X}); S(K(\cdot, x))(1) < \infty \}. \tag{1}$$

Observe that $\sigma(\bar{X})$ is a linear subspace of $\Sigma(\bar{X})$ which contains all the spaces $(\bar{X})_{\theta, q}$.

A pair of operators $\Omega_{\bar{X}} : \sigma(\bar{X}) \to \Sigma(\bar{X})$, $\Omega_{\bar{Y}} : \sigma(\bar{Y}) \to \Sigma(\bar{Y})$ will be said to be *K-commuting* if there exists a constant $C > 0$ such that

$$K(t, [T, \Omega](x)) \leqslant C \, \|T\| \, S(K(\cdot, x))(t). \tag{2}$$

for any $x \in \sigma(\bar{X})$ and $T \in \mathscr{L}(\bar{X}; \bar{Y})$. Here $[T, \Omega] = T\Omega_{\bar{X}} - \Omega_{\bar{Y}} T$.

In this case, Ω is well defined on the spaces $(\bar{X})_{\theta, q}$ and for all $T \in \mathscr{L}(\bar{X}; \bar{Y})$ we obtain from (2) the commutator theorem

$$\|[T, \Omega](x)\|_{\theta, q} \leqslant c \, \|T\| \, \|x\|_{\theta, q}, \tag{3}$$

with $c > 0$ independent of $T \in \mathscr{L}(\bar{X}; \bar{Y})$ and $x \in (\bar{X})_{\theta, q}$. See the proof of Theorem (4.3) in [MS].

We refer to [BK], [BL], and [BS] for general results concerning interpolation theory.

In the main result below, Banach couples are pairs of Besov spaces. For simplicity, we only consider L_p-approximation by trigonometric polynomials, and by $B_p^{\sigma, q} = B^{\sigma, q}(L_p)$ we denote the Besov space of 2π-periodic complex-valued functions (see [BB] and [DL]). Its norm satisfies

$$\|f\|_{B_p^{\sigma, q}} \simeq \left(\sum_{n=0}^{\infty} [2^{n\sigma} e_{2^n}(f)]^q \right)^{1/q}, \tag{4}$$

where $e_{2^n}(f) = d(f, A_{2^n})$ is the L_p-distance from f to the space A_{2^n} of all trigonometric polynomials

$$\sum_{|k| < 2^{n-1}} c_k e^{ikx}$$

of degree less than 2^{n-1} (and $A_1 = \{0\}$).

As usual, $F \simeq G$ means that $F \leqslant C_1 G$ and $G \leqslant C_2 F$ for some constants C_1 and C_2.

Let $\mu = \{\mu_k\}_{k=-\infty}^{+\infty}$ be a sequence of complex numbers. We shall consider the multiplier operator defined on the trigonometric polynomials by

$$T_\mu\left(\sum_k c_k e^{ikx}\right) = \sum_k \mu_k c_k e^{ikx}.$$

Our main result is the following theorem whose proof will be given in Section 4. If $n \in \mathbf{N}$, we denote $\Delta_n = \{k \in \mathbf{Z};\, 2^{n-1} \leqslant k < 2^n\}$, $\Delta_{-n} = -\Delta_n$, and $\Delta_0 = \{0\}$.

THEOREM 1. *Suppose that* $\mu = \{\mu_k\}_{k=-\infty}^{+\infty}$ *is a sequence of complex numbers such that*

$$\sup_k |\mu_k - \mu_{-k}| = c_1 < \infty \qquad (\mu_{-0} = 0) \tag{5}$$

and

$$\sup_{n \in \mathbf{Z}} \left(\sum_{k \in \Delta_n} |\mu_{k+1} - \mu_k| \right) = c_2 < \infty. \tag{6}$$

Then the operator T_μ *is K-commuting* (*see* (2)) *for the couples* $(B_p^{\sigma_0, q_0}, B_p^{\sigma_1, q_1})$ $(1 < p < \infty,\ \sigma_0 > \sigma_1,\ q_0, q_1 \geqslant 1)$.

The constant c in (2) *can be estimated by* $c \leqslant \gamma \max(c_1, c_2)$, *with* γ *a constant which depends only on the parameters* σ *and* q.

Thus, for $\sigma_0 > \sigma_1 > 0$,

$$\|[T, T_\mu]\|_{B_p^{\sigma, q}, B_p^{\sigma, q}} \leqslant c \|T\|,$$

where $\|T\| = \max(\|T\|_{B_p^{\sigma_0, q_0}, B_p^{\sigma_0, q_0}}, \|T\|_{B_p^{\sigma_1, q_1}, B_p^{\sigma_1, q_1}})$, $\sigma = (1-\theta)\,\sigma_0 + \theta\sigma_1$ for some $\theta \in (0, 1)$, and $q_0, q_1 \geqslant 1$. Under these conditions it is known (cf. [BL]) that

$$(B_p^{\sigma_0, q_0}, B_p^{\sigma_0, q_0})_{\theta, q} = B_p^{\sigma, q}.$$

Remark 1. Simple examples, such as the operator $Tf(x)=f(-x)$, show that condition (5) is necessary. Moreover, if the sequence $\{\mu_k\}$ is diadic, in the sense that it is constant in every interval Δ_n, condition (6) has the form

$$\sup_n\{\max(|\mu_{2^{n+1}}-\mu_{2^n}|, |\mu_{-2^{n+1}}-\mu_{-2^n}|)\}<\infty$$

and it is also necessary (e.g., consider the operator $Tf(x)=f(2x)$).

Remark 2. In our Theorem 1, which is independent of the Marcinkiewicz theorem, the conditions on μ are much weaker than those of that theorem. In our case sequences as $\mu_k=\log|k|$ $(k=\pm 1, \pm 2, ...)$ are allowed.

Recall (cf. [EG]) that the strong form of the Marcinkiewicz theorem states that, under the conditions (6) and

$$\sup_k |\mu_k|<\infty,$$

T_μ is an L_p-multiplier $(1<p<\infty)$.

This paper is organized as follows:

In Section 2 we prove a commutator theorem (3) which applies to certain operators of the type

$$\Omega(x)=\sum_{j<0}\lambda_j x_0(t_j)+\sum_{j\geqslant 0}\lambda_j x_1(t_j),$$

associated to decompositions $x=x_0(t)+x_1(t)$ which are almost optimal for the K-functional, in the sense that $K(t, x)\simeq \|x_0(t)\|_{X_0}+t\|x_1(t)\|_{X_1}$. This result can be considered special instance of the method given in [CCS], and extends the results of [JRW], and the real version of the previous construction in [RW] for the complex method; it is also related to [MS; Theorem (3.7)].

Section 3 deals with approximation spaces. A "Holmstedt-type formula" for the K-functional provides an almost optimal decomposition for the K-functional of these spaces.

In Section 4 the previous results are applied to the commutators for multipliers on Besov spaces by representing them as approximation spaces.

2. THE COMMUTATOR THEOREM

For a given Banach couple $\bar{X}$, let $H(\bar{X})$ be the Banach space of all measurable functions

$$(x_0, x_1)\colon \mathbf{R}^+\to X_0\times X_1$$

such that $x_0(t)+x_1(t)=\Phi(x_0,x_1)\in\Sigma(\bar X)$, constant, and

$$\|(x_0,x_1)\|_H=S(\alpha(x_0,x_1))(1)<\infty,$$

where $\alpha(x_0,x_1)(t)=\|x_0(t)\|_0+t\|x_1(t)\|_1$.

The operator $\Phi_{\bar X}=\Phi : H(\bar X)\to\Sigma(\bar X)$, such that $\Phi(x_0,x_1)=x_0(t)+x_1(t)$, is bounded, since

$$\|\Phi(x_0,x_1)\|_\Sigma=\|x_0(t)+x_1(t)\|_\Sigma\leqslant 2\int_1^2\alpha(x_0,x_1)(s)\,\frac{ds}{s}\leqslant 2\,\|(x_0,x_1)\|_H.$$

If $\bar Y$ is a second Banach couple and $T\in\mathscr{L}(\bar X;\bar Y)$, we define $H(T)(x_0,x_1)=(Tx_0,Tx_1)$ and we obtain a linear operator $H(T) : H(\bar X)\to H(\bar Y)$ such that $\|H(T)\|\leqslant\|T\|_{\bar X;\bar Y}$ and $T\circ\Phi_{\bar X}=\Phi_{\bar Y}\circ H(T)$.

Observe that $\sigma(\bar X)$ is the image space, $\Phi(H(\bar X))$, endowed with the quotient norm

$$\|x\|_\Phi=\inf_{x=x_0(t)+x_1(t)}\|(x_0,x_1)\|_H=S(K(\cdot,x))(1),$$

since obviously $S(K(\cdot,x))(1)\leqslant\|x\|_\Phi$ and, on the other hand, if $x\in\sigma(\bar X)$, we can consider $x=x_0(t)+x_1(t)$ such that $\alpha(x_0,x_1)(t)\leqslant(1+\varepsilon)\,K(t,x)$ and then $(x_0,x_1)\in H(\bar X)$ with $\|x\|_\Phi\leqslant S(\alpha(x_0,x_1))(1)\leqslant(1+\varepsilon)\,S(K(\cdot,x))(1)$.

For every Banach couple $\bar X$, let $\Psi_{\bar X}=\Psi : H(\bar X)\to\Sigma(\bar X)$ be a second operator such that $T\circ\Psi_{\bar X}=\Psi_{\bar Y}\circ H(T)$.

Let $c>1$, a fixed constant. If for every $x\in\sigma(\bar X)$ we choose an *almost optimal decomposition* for the K-functional, $h_x=(x_0,x_1)$, in the sense that

$$x_0(t)+x_1(t)=x \qquad\text{and}\qquad \alpha(x_0,x_1)(t)\leqslant cK(t,x),$$

then $\|h_x\|_H\leqslant c\,\|x\|_\Phi$.

The associated operator $\Omega_{\bar X}=\Omega$ will be the operator, generally non-linear, defined by

$$\Omega(x)=\Psi(h_x).$$

The following lemma, which is an abstract version of the commutator theorem, shows where cancellation takes place.

Lemma 1. *Assume that Ψ satisfies the following condition: For every $(x_0,x_1)\in H(\bar X)$ such that $x_0+x_1=0$, there exists a measurable function*

$$(y_0,y_1)\colon \mathbf{R}^+\to X_0\times X_1$$

with the properties

$$y_0(t)+y_1(t)=\Psi(x_0,x_1) \quad \textit{and} \quad \alpha(y_0,y_1)(t)\leqslant cS(\alpha(x_0,x_1))(t) \textit{ for all } t>0,$$

where c is a constant which does not depend on (x_0, x_1).

Then Ω *is K-commuting; i.e.,*

$$K(t,[T,\Omega](x))\leqslant C\,\|T\|\,S(K(\cdot,x))(t).$$

Proof. Let $x\in\sigma(\bar{X})$. Then $Tx\in H(\bar{Y})$, and for the almost optimal decompositions $h_x\in H(\bar{X})$ and $h_{Tx}\in H(\bar{Y})$ we have $\alpha(h_x)(t)\leqslant cK(t,x)$ and $\alpha(h_{Tx})(t)\leqslant cK(t,Tx)\leqslant c\,\|T\|\,K(t,x)$.

Then

$$[T,\Omega]\,x=T\Psi h_x-\Psi h_{Tx}=\Psi_{\bar{Y}}\,(H(T)\,h_x-h_{Tx})$$

with $H(T)\,h_x-h_{Tx}\in H(\bar{Y})$ and $\Phi(H(T)\,h_x-h_{Tx})=0$. Hence, there exists $(y_0(t),y_1(t))$ such that $y_0+y_1=\Psi(H(T)\,h_x-h_{Tx})$ and $\alpha(y_0,y_1)(t)\leqslant cS(\alpha(H(T)\,h_x-h_{Tx}))(t)$. Thus

$$K(t,[T,\Omega]\,x)\leqslant\alpha(y_0,y_1)(t)\leqslant cS(\alpha(H(T)\,h_x-h_{Tx}))(t).$$

To estimate the right-hand side, we observe that $\alpha(H(T)\,h_x-h_{Tx})\leqslant 2c\,\|T\|\,K(t,x)$, and S is positive. ▌

We associate to every $\lambda\in L_\infty(\mathbf{R}^+)$ the operator $\Psi_{\bar{X}}\colon H(\bar{X})\to\Sigma(\bar{X})$ such that

$$\Psi_{\bar{X}}\,(x_0,x_1)=\int_0^1\lambda(t)\,x_0(t)\,\frac{dt}{t}+\int_1^\infty\lambda(t)\,x_1(t)\,\frac{dt}{t},$$

and, for a given almost optimal decomposition, the corresponding operator $\Omega_{\bar{X}}=\Omega$.

The operator $\Psi_{\bar{X}}$ is bounded, since

$$\left\|\int_0^1\lambda(t)\,x_0(t)\,\frac{dt}{t}\right\|_0\leqslant\|\lambda\|_\infty\int_0^1\|x_0(t)\|_0\,\frac{dt}{t}\leqslant\|\lambda\|_\infty\int_0^1\alpha(x_0,x_1)(t)\,\frac{dt}{t},$$

and similarly

$$\left\|\int_1^\infty\lambda(t)\,x_1(t)\,\frac{dt}{t}\right\|_1\leqslant\|\lambda\|_\infty\int_1^\infty\alpha(x_0,x_1)(t)\,\frac{dt}{t^2},$$

thus, $\|\Psi_{\bar{X}}(x_0,x_1)\|_\Sigma\leqslant\|\lambda\|_\infty\,\|(x_0,x_1)\|_H$.

THEOREM 2. *For every $\lambda \in L_\infty(\mathbf{R}^+)$, the associated operator Ω is K-commuting for all Banach couples $\bar{X}$.*

Proof. Let $(x_0, x_1) \in H(\bar{X})$ as in Lemma 1. Then, since $x_1(t) = -x_0(t) \in X_0 \cap X_1$,

$$\Psi(x_0, x_1) = \int_0^1 \lambda(t)\, x_0(t) \frac{dt}{t} - \int_1^\infty \lambda(t)\, x_0(t) \frac{dt}{t},$$

and, if $\Psi(x_0, x_1) = y_0(t) + y_1(t)$ is an almost optimal decomposition for the K-functional, it follows that

$$\alpha(y_0, y_1)(t) \leqslant cK(t, \Psi(x_0, x_1)) \leqslant c \int_0^\infty |\lambda(s)|\, J(s, x_0(s)) \min\left(1, \frac{t}{s}\right) \frac{ds}{s}$$
$$\leqslant c\, \|\lambda\|_\infty\, S(\alpha(x_0, x_1))(t). \quad \blacksquare$$

Let us now denote $D_n = [2^{n-1}, 2^n)$ $(n \in \mathbf{Z})$, the dyadic intervals of $(0, \infty)$.

THEOREM 3. *Let $\{t_j\}_{j \in \mathbf{Z}} \subset (0, \infty)$ be an increasing sequence such that $t_j \leqslant 1$ if $j < 0$ and $t_j \geqslant 1$ if $j \geqslant 0$, $t_j \uparrow \infty$ as $j \uparrow +\infty$ and $t_j \downarrow 0$ as $j \downarrow -\infty$, and $\{\lambda_j\}_{j \in \mathbf{Z}}$ any sequence of complex numbers such that*

$$M^* = \sup_n \sum_{t_j \in D_n} |\lambda_j| < \infty. \tag{7}$$

For a given Banach couple, $\bar{X}$, and for every t_j let $x = x_0(t_j) + x_1(t_j)$ be a decomposition such that

$$\|x_0(t_j)\|_0 + t_j\, \|x_1(t_j)\|_1 \leqslant cK(t_j, x) \qquad (x \in \sigma(\bar{X})),$$

where $c > 1$ is a constant.

Then

$$\Omega(x) = \sum_{j<0} \lambda_j x_0(t_j) + \sum_{j \geqslant 0} \lambda_j x_1(t_j)$$

defines a K-commuting operator on $\bar{X}$.

Proof. If $\lambda_{D_n} = \sum_{t_j \in D_n} \lambda_j$, the operator

$$\Omega^*(x) = \sum_{n<0} (\lambda_{D_n} - 1)\, x_0(2^n) + \sum_{n \geqslant 0} (\lambda_{D_n} - 1)\, x_1(2^n),$$

with $x = x_0(2^n) + x_1(2^n)$ and $\|x_0(2^n)\|_0 + 2^n \|x_1(2^n)\|_1 \leqslant cK(2^n, x)$, is K-commuting, since

$$\Omega^*(x) = \frac{1}{\log 2}\left(\sum_{n<0} \int_{D_n} (\lambda_{D_n} - 1)\, x_0(2^n)\, \frac{dt}{t} + \sum_{n \geqslant 0} \int_{D_n} (\lambda_{D_n} - 1)\, x_1(2^n)\, \frac{dt}{t}\right)$$
$$= \frac{1}{\log 2}\left(\int_0^1 \lambda(t)\, x_0(t)\, \frac{dt}{t} + \int_1^\infty \lambda(t)\, x_1(t)\, \frac{dt}{t}\right)$$

for $\lambda(t) = \lambda_{D_n} - 1$ and $x_i(t) = x_i(2^n)$ if $t \in D_n$, and Theorem 2 applies.

We only need to prove that the operator

$$(\Omega - \Omega^*)\, x = \sum_{n<0} \left[\sum_{t_j \in D_n} \lambda_j x_0(t_j) - (\lambda_{D_n} - 1)\, x_0(2^n)\right] + \sum_{n \geqslant 0} \left[\sum_{t_j \in D_n} \lambda_j x_1(t_j) - (\lambda_{D_n} - 1)\, x_1(2^n)\right]$$

is K-commuting.

For every D_n we have the decomposition

$$\sum_{t_j \in D_n} \lambda_j x_0(t_j) - (\lambda_{D_n} - 1)\, x_0(2^n) + \sum_{t_j \in D_n} \lambda_j x_1(t_j) - (\lambda_{D_n} - 1)\, x_1(2^n) = x$$

and

$$\left\| \sum_{t_j \in D_n} \lambda_j x_0(t_j) - (\lambda_{D_n} - 1)\, x_0(2^n) \right\|_0$$
$$\leqslant M^* \max_{t_j \in D_n} \|x_0(t_j)\|_0 + (M^* + 1) \|x_0(2^n)\|_0$$
$$\leqslant cM^* K(2^{n+1}, x) + c(M^* + 1)\, K(2^n, x)$$
$$\leqslant c(3M^* + 1)\, K(2^n, x).$$

Similarly, $\|\sum_{t_j \in D_n} \lambda_j x_1(t_j) - (\lambda_{D_n} - 1)\, x_1(2^n)\|_1 \leqslant c(2M^* + 1)\, K(2^n, x)/2^n$.

It follows that the above decomposition is almost optimal and, in the same way as for Ω^*, the operator $\Omega - \Omega^*$ is K-commuting. ∎

We associate to $\{\lambda_j\}$ the new sequence $\{\mu_j\}$, with

$$\mu_k = \begin{cases} \sum_{k \leqslant j < 0} \lambda_j & \text{if } k < 0, \\ \sum_{0 \leqslant j < k} \lambda_j & \text{if } k > 0, \\ \lambda_0 & \text{if } k = 0. \end{cases}$$

Thus $|\lambda_k| = |\mu_{k+1} - \mu_k|$ if $k \neq 0$, and the condition (7) is equivalent to

$$\sup_{n \in \mathbf{Z}} \sum_{t_k \in D_n} |\mu_{k+1} - \mu_k| < \infty. \tag{8}$$

For every $x \in \sigma(\bar{X})$ we have $\lim_{t \to 0} K(x, t) = \lim_{t \to \infty} K(x, t)/t = 0$ and, if we denote $u_k = x_0(t_k) - x_0(t_{k-1})$, in $\Sigma(\bar{X})$ we have $x_0(t_j) = \sum_{k \leqslant j} u_k$ and $x_1(t_j) = x - x_0(t_j) = \sum_{k > j} u_k$, and then

$$\begin{aligned}\Omega x &= \sum_{j<0} \lambda_j \left(\sum_{k \leqslant j} u_k \right) + \sum_{j \geqslant 0} \lambda_j \left(\sum_{k > j} u_k \right) \\ &= \sum_{k<0} \left(\sum_{k \leqslant j < 0} \lambda_j \right) u_k + \sum_{k \geqslant 0} \left(\sum_{0 \leqslant j < k} \lambda_j \right) u_k.\end{aligned}$$

Hence we can describe the Ω-operator as

$$\Omega x = \sum_{k<0} \mu_k (x_0(t_k) - x_0(t_{k-1})) + \sum_{k \geqslant 0} \mu_k (x_0(t_k) - x_0(t_{k-1}))$$

and it is K-commuting if condition (8) is satisfied.

Corollary 1. *Let $\{t_j\}$ be a sequence as in Theorem 3 and $\{\mu_j\}$ be a sequence of complex numbers such that*

$$C^* = \sup_{n \in \mathbf{Z}} \sum_{t_k \in D_n} |\mu_{k+1} - \mu_k| < \infty. \tag{9}$$

Then

$$\Omega(x) = \sum_{k \in \mathbf{Z}} \mu_k (x_0(t_{k+1}) - x_0(t_k))$$

defines a K-commuting operator for any Banach couple.

Remark 3. It is easily seen (cf. [EG]) that, for any $\delta > 1$, condition (9) is equivalent to

$$C^*_\delta = \sup_{n \in \mathbf{Z}} \sum_{t_k \in [\delta^n, \delta^{n+1})} |\mu_{k+1} - \mu_k| < \infty. \tag{10}$$

3. ALMOST OPTIMAL DECOMPOSITION FOR APPROXIMATION SPACES

Let $\mathscr{V}$ be a Hausdorff topological linear space and X be a Banach subspace of $\mathscr{V}$, with continuous embedding $X \subsetneq \mathscr{V}$.

Let us also consider a fixed *approximation family* A_t $(t>0)$, which is a family of nonempty subsets of $\mathscr{V}$ with the following properties:

(a) $A_s \subset A_t$ if $s<t$,

(b) $-A_t = A_t$, and

(c) $A_s + A_t \subset A_{s+t}$.

It is clear that $0 \in \bigcap_{t>0} A_t$ and that $A = \bigcup_{t>0} A_t$ is an abelian group that will be endowed with the (semi-)norm

$$\|x\|_A = \inf\{t>0;\, x \in A_t\}.$$

Then, as in [PS], we can define the *approximation spaces* $E_{p,q}$, similar to the Lorentz spaces $L_{p,q}$, of all elements $f \in A+X$ such that

$$\|f\|_{E_{p,q}} = \left(\int_0^\infty [t^{1/p} E(f,t)]^q \frac{dt}{t}\right)^{1/q} < \infty,$$

with $E(f,t) = \inf_{a \in A_t} \|f-a\|_X$. By f_t we denote an element in A_t such that

$$\|f - f_t\|_X \leqslant c E(f,t) \tag{11}$$

with $c>1$ independent of $t>0$ and f.

A typical example (see [PS] or [Ni]) appears for $\mathscr{V} = L_0$, the space of all measurable functions on $\mathbf{R}^n$, $X = L_\infty$ and $A_t = \{f \in L_0;\, \|f\|_0 = |\operatorname{supp} f| \leqslant t\}$. In this case

$$E(f,t) = f^*(t),$$

the nonincreasing rearrangement of f, $E_{p,q} = L_{p,q}$ and we have the Holmstedt formula for couples of Lorentz spaces,

$$K(t^{1/p_0 - 1/p_1}, f; L_{p_0,q_0}, L_{p_1,q_1}) \simeq \left(\int_0^t (s^{1/p_0} f^*(s))^{q_0} \frac{ds}{s}\right)^{1/q_0} + t^\delta \left(\int_t^\infty (s^{1/p_1} f^*(s))^{q_1} \frac{ds}{s}\right)^{1/q_1},$$

to estimate the K-functional.

A similar result holds for couples of approximation spaces and gives an estimate for the K-functional:

THEOREM 4. *If $(E_{p_0,q_0}, E_{p_1,q_1})$ is a couple of approximation spaces and $p_0 < p_1$, then*

(a) $K(t^{1/p_0-1/p_1}, f; E_{p_0,q_0}, E_{p_1,q_1}) \simeq \|f_t\|E_{p_0,q_0} + t^{1/p_0-1/p_1}\|f-f_t\|E_{p_1,q_1}$,

and

(b) $K(t^{1/p}, f; E_{p,q}, X) \simeq \|f_t\|_{E_{p,q}} + t^{1/p}\,\|f-f_t\|_X$.

Proof. (a) Let $\delta = 1/p_0 - 1/p_1$. It is known (cf. [Ni]) that

$$K(t^\delta, f; E_{p_0,q_0}, E_{p_1,q_1}) \simeq \left(\int_0^t (s^{1/p_0}E(f,s))^{q_0}\frac{ds}{s}\right)^{1/q_0} + t^\delta\left(\int_t^\infty (s^{1/p_1}E(f,s))^{q_1}\frac{ds}{s}\right)^{1/q_1}. \tag{12}$$

If f_t is as in (11), we have $E(f_t, s) = 0$ when $s > t$ and $E(f_t, s) \leqslant 2cE(f, s)$ when $s \leqslant t$, since $\|f_t - f_s\| \leqslant cE(f,t) + cE(f,s) \leqslant 2cE(f,s)$. Hence

$$\|f_t\|E_{p_0,q_0} = \left(\int_0^t \left(s^{1/p_0}E(f_t,s)\right)^{q_0}\frac{ds}{s}\right)^{1/q_0} \leqslant 2c\left(\int_0^t \left(s^{1/p_0}E(f,s)\right)^{q_0}\frac{ds}{s}\right)^{1/q_0}. \tag{13}$$

On the other hand,

$$t^\delta\,\|f-f_t\|E_{p_1,q_1} = t^\delta\left(\int_0^\infty (\,s^{1/p_1}E(f-f_t,s))^{q_1}\frac{ds}{s}\right)^{1/q_1} = I_1 + I_2$$

with

$$I_1 = t^\delta\left(\int_0^{2t} (s^{1/p_1}E(f-f_t,s))^{q_1}\frac{ds}{s}\right)^{1/q_1} \quad \text{and}$$

$$I_2 = t^\delta\left(\int_{2t}^\infty (s^{1/p_1}E(f-f_t,s))^{q_1}\frac{ds}{s}\right)^{1/q_1}.$$

From $E(f-f_t, s) \leqslant \|f-f_t\|_X$ we obtain the estimate

$$I_1 \leqslant (p_1/q_1)^{1/q_1}\,t^\delta\,\|f-f_t\|_X\,2^{1/p_1}\,t^{1/p_1} \leqslant c(p_1/q_1)^{1/q_1}\,2^{1/p_1}t^{1/p_0}\,E(f,t)$$

$$\leqslant c(p_1/q_1)^{1/q_1}\,2^{1/p_1}\left(\int_0^t (s^{1/p_0}E(f,s))^{q_0}\frac{ds}{s}\right)^{1/q_0}$$

and, since in $E(f-f_t, s) \leqslant E(f, s/2) + E(f_t, s/2)$, $E(f_t, s/2) = 0$ when $s \geqslant 2t$,

$$I_2 \leqslant t^\delta \left(\int_{2t}^\infty (s^{1/p_1} E(f, s/2))^{q_1} \frac{ds}{s} \right)^{1/q_1}$$

$$= t^\delta \left(\int_t^\infty ((2s)^{1/p_1} E(f, s))^{q_1} \frac{ds}{s} \right)^{1/q_1}.$$

By combining these estimates, (13) and (12) it follows that

$$\|f_t\|E_{p_0, q_0} + t^\delta \|f - f_t\|E_{p_1, q_1} \leqslant CK(t^\delta, f; E_{p_0, q_0}, E_{p_1, q_1}).$$

Obviously, $K(t^\delta, f; E_{p_0, q_0}, E_{p_1, q_1}) \leqslant \|f_t\|E_{p_0, q_0} + t^\delta \|f - f_t\|E_{p_1, q_1}$.

The proof of (b) is the same, starting now from

$$K(t^{1/p}, f; E_{p, q}, X) \simeq \left(\int_0^t (s^{1/p} E(f, s))^q \frac{ds}{s} \right)^{1/q}$$

instead of (12). ∎

4. PROOF OF THEOREM 1

Proof. Let us denote by $A_{n, m}$ ($n < m$ in $\mathbf{Z}$) the linear space of trigonometric polynomials with basis e^{ikx} ($n < k < m$), and by $P_{n, m}$ the projection

$$P_{n, m} \left(\sum_{k=-\infty}^{+\infty} c_k e^{ikx} \right) = \sum_{n<k<m} c_k e^{ikx}$$

from L_p onto $A_{n, m}$. If $p \in (1, \infty)$, we shall use the uniform boundedness on L_p of these projections,

$$\sup_{n<m} \|P_{n, m}\|_{L_p, L_p} < \infty, \tag{14}$$

which follows from the boundedness properties of the Hilbert transform.

To show that the Besov space $B_p^{\sigma, q}$ is an approximation space, $E_{1/\sigma, q}$, we shall construct an approximation family of linear spaces A_t ($t > 0$).

For $t = 2^n$, $n \in \mathbf{N}$, A_t is the space, defined in Section 1, of all trigonometric polynomials of the type

$$\sum_{|k| < 2^{n-1}} c_k e^{ikx},$$

with $A_1 = \{0\}$.

To define A_N when $N \in \mathbf{N}$ and $2^n < N < 2^{n+1}$, we take as generators the basis of A_{2^n} plus the first $N - 2^n$ terms of the sequence

$$e^{-2^{n-1}xi}, e^{-(2^{n-1}+1)xi}, ..., e^{-(2^n-1)xi}, e^{2^{n-1}xi}, e^{(2^{n-1}+1)xi}, ..., e^{(2^n-1)xi}.$$

For $0 < t \notin \mathbf{Z}$ we define $A_t = A_{[t]+1}$.

The dimension of A_N is $N-1$, it is of type $A_{n,m}$ and the projections

$$P_t = P_{n,m}: L_p \to A_t$$

are uniformly bounded.

Now we can consider the space $E_{p,q}$ with the norm

$$\|f\|_{E_{p,q}} = \left(\int_0^\infty [t^{1/p} E(f, t)]^q \frac{dt}{t}\right)^{1/q}.$$

We have

$$e_{2^{n+1}}(f) \leqslant E(f, t) \leqslant e_{2^n}(f) \qquad (2^n \leqslant t \leqslant 2^{n+1}),$$

and, since $A_t = \{0\}$ for $t < 1$,

$$\|f\|_{E_{p,q}} \simeq \left(\sum_{n=0}^{\infty} [2^{n/p} e_{2^n}(f)]^q\right)^{1/q}.$$

Thus, by (4),

$$\|f\|_{E_{1/\sigma, q}} \simeq \|f\|_{B_p^{\sigma, q}}$$

and our couple $(B_p^{\sigma_0, q_0}, B_p^{\sigma_1, q_1})$ is just $(E_{1/\sigma_0, q_0}, E_{1/\sigma_1, q_1})$.

Moreover, from the uniform boundedness of the projections P_t it follows that

$$\|f - P_t f\|_{L_p} \leqslant cE(f, t)$$

with $c \geqslant 1$ independent of $t > 0$ and f. By Theorem 4, this means that $(P_t f, f - P_t f)$ is an almost optimal decomposition of f for the couple $(B_p^{\sigma_0, q_0}, B_p^{\sigma_1, q_1})$ and that for all $t > 0$ we have

$$K(t^{\sigma_0 - \sigma_1}, f; B_p^{\sigma_0, q_0}, B_p^{\sigma_1, q_1}) \simeq \|P_t f\|_{B_p^{\sigma_0, q_0}} + t^{\sigma_0 - \sigma_1} \|f - P_t f\|_{B_p^{\sigma_1, q_1}}.$$

We shall consider the sequence $t_n = n^{\sigma_0 - \sigma_1}$ if $n = 1, 2, ...$ (and any $t_n \downarrow 0$ as $n \downarrow -\infty$), and the operator

$$\Omega: \sigma(B_p^{\sigma_0, q_0}, B_p^{\sigma_1, q_1}) \to B_p^{\sigma_0, q_0} + B_p^{\sigma_1, q_1}$$

defined by

$$\Omega(f)=\sum_{k=1}^{\infty}\mu_k^*(P_{k+1}f-P_kf),$$

where $\{\mu_k^*\}_{k=1}^{\infty}$ is the rearrangement of $\{\mu_j\}_{j=-\infty}^{+\infty}$ defined as follows. If e^{in_kx} is the element added to the basis of A_k to obtain A_{k+1}, then $\mu_k^*=\mu_{n_k}$ $(k\geqslant 1)$. Set $\mu_k^*=0$ if $k\leqslant 0$.

For $f=\sum_k c_k e^{ikx}$, since

$$[(P_{k+1}-P_k)f](x)=c_{n_k}e^{in_kx},$$

we have that $T_\mu(f)=\Omega(f)$.

The sequence $\{\mu_k^*\}_{k=1}^{\infty}$ satisfies condition (9) of the commutator theorem or the equivalent property (10). This follows from (5) and (6), since

$$|\mu_{-(2^{n+1}-1)}-\mu_{2^n}|\leqslant|\mu_{-(2^{n+1}-1)}-\mu_{-2^n}|+|\mu_{-2^n}-\mu_{2^n}|\leqslant c_1+c_2$$

and then, if $\delta=2^{\sigma_0-\sigma_1}$,

$$\begin{aligned}\sum_{t_k\in[\delta^n,\,\delta^{n+1})}&|\mu_{k-1}^*-\mu_k^*|\\ &\simeq|\mu_{2^{n+1}-1}-\mu_{2^{n+1}-2}|+|\mu_{2^{n+1}-2}-\mu_{2^{n+1}-3}|\\ &\quad+\cdots+|\mu_{2^n+1}-\mu_{2^n}|\\ &\quad+|\mu_{2^n}-\mu_{-(2^{n+1}-1)}|\\ &\quad+|\mu_{-(2^{n+1}-2)}-\mu_{-(2^{n+1}-3)}|+\cdots+|\mu_{-(2^n+1)}-\mu_{-2^n}|\\ &\leqslant C.\end{aligned}$$

An application of Corollary 1 ends the proof.

REFERENCES

[BB] P. L. Butzer and H. Berens, "Semi-Groups of Operators and Approximation," Springer-Verlag, New York, 1967.

[BK] Yu. A. Brudnyiĭ and N. Ya. Krugljak, "Interpolation Functors and Interpolation Spaces," North Holland, Amsterdam/New York/Oxford/Tokyo, 1991.

[BL] J. Bergh and J. Löfström, "Interpolation Spaces," Springer-Verlag, New York, 1976.

[BS] C. Bennett and B. Sharpley, "Interpolation of Operators," Academic Press, New York, 1988.

[CCS] M. J. Carro, J. Cerdà, and J. Soria, Commutators and interpolation methods, *Ark. Mat.* **33** (1995), 199–216.

[DL] R. A. DeVore and G. G. Lorentz, "Constructive Approximation," Springer-Verlag, Berlin/Heidelberg/New York, 1993.

[EG] R. E. Edwards and G. I. Gaudry, "Littlewood–Paley and Multiplier Theory," Springer-Verlag, Berlin/Heidelberg/New York, 1977.

[MS] M. Milman and T. Schonbek, Second order estimates in interpolation theory and applications, *Proc. Amer. Math. Soc.* **110** (1990), 961–969.

[JRW] B. Jawerth, R. Rochberg, and G. Weiss, Commutators and other second order estimates in real interpolation theory, *Ark. Mat.* **24** (1986), 191–219.

[Ni] P. Nilsson, Reiteration theorems for real interpolation and approximation spaces, *Ann. Math. Pura Appl.* **132** (1982), 291–330.

[PS] J. Peetre and G. Sparr, Interpolation of normed abelian groups, *Ann. Math. Pura Appl.* **92** (1972), 217–262.

[RW] R. Rochberg and G. Weiss, Derivatives of analytic families of Banach spaces, *Ann. of Math.* **118** (1983), 315–347.